Maschke's theorem and irreducible representations for skew braces

Cindy Tsang

Ochanomizu University

Joint work with Yuta Kozakai.

arXiv:2405.08662

This work is supported by JSPS KAKENHI 24K16891.

Hopf Algebras and Galois Module Theory

Online & University of Nebraska Omaha

2024/5/28 $\sim$ 2024/5/31

Introduction

Skew brace

- Skew brace is an algebraic structure that was introduced as a tool to study the non-degenerate set-theoretic solutions to the Yang-Baxter equation.

Definition (Guarnieri-Vendramin 2017)

A *skew brace* is a set $A = (A, \cdot, \circ)$ equipped with two operations such that

1. $(A, \cdot)$ is a group (called the additive group of the skew brace A);
2. $(A, \circ)$ is a group (called the multiplicative group of the skew brace A);
3. $a \circ (b \cdot c) = (a \circ b) \cdot a^{-1} \cdot (a \circ c)$ holds for all $a, b, c \in A$.

- **Note.** It is easy to see that $(A, \cdot)$ and $(A, \circ)$ share the same identity.
- **Example.** If $G = (G, \cdot)$ is a group, then
  1. $(G, \cdot, \cdot)$ is a skew brace; (trivial skew brace)
  2. $(G, \cdot, \cdot^{\mathrm{op}})$ is also a skew brace. (almost trivial skew brace)

Thus, in some sense, skew brace can be viewed as an extension of group.

Similarities with groups

- Can concepts and techniques from group theory be extended to skew braces?
- Many concepts in groups admit analogs in skew braces.
  a. center: socle, annihilator, etc
  b. nilpotent: left nilpotent, right nilpotent, centrally nilpotent, etc
- There is a trend to try and extend theories in groups to skew braces.
  1. Factorization of skew braces [Jespers *et al.* 2019, T. 2024]
     - There is analog of Ito's theorem for skew braces.
  2. Isoclinism of skew braces [Letourmy-Vendramin 2023, Caranti 2024]
     - Every skew brace is isoclinic to a so-called stem skew brace.
  3. Schur cover of skew braces [Letourmy-Vendramin 2024]
     - The Schur cover of a finite skew brace is unique up to isoclinism.
- In relation to Schur cover, they also introduced (complex projective) representation of skew braces, whose definition was based of [Zhu 2020].

Representation of a skew brace

Definition

- Let $A = (A, \cdot, \circ)$ be a skew brace.

Definition (Letourmy-Vendramin 2024/Zhu 2020)

A *representation* of A over a field k is a triplet (V, β, ρ), where

1. V is a vector space over k;
2. $\beta : (A, \cdot) \longrightarrow \mathrm{GL}(V)$ is a representation of the additive group;
3. $\rho : (A, \circ) \longrightarrow \mathrm{GL}(V)$ is a representation of the multiplicative group;

such that the compatibility relation

$$\rho(a)\beta(b)\rho(a)^{-1} = \beta((a \circ b) \cdot a^{-1})$$

holds for all $a, b, c \in A$.

- Where did this compatibility relation come from?

The compatibility relation

- Let $A = (A, \cdot, \circ)$ be a skew brace.
- The so-called lambda homomorphism of A is defined by

$$\lambda : (A, \circ) \longrightarrow \mathrm{Aut}(A, \cdot); \ \ \lambda_a(b) = a^{-1} \cdot (a \circ b).$$

Analogously, we can define a homomorphism

$$\lambda^{\mathrm{op}} : (A, \circ) \longrightarrow \mathrm{Aut}(A, \cdot); \ \ \lambda^{\mathrm{op}}_a(b) = (a \circ b) \cdot a^{-1}.$$

- This gives rise to an outer semidirect product

$$(A, \cdot) \rtimes_{\lambda^{\mathrm{op}}} (A, \circ).$$

In this group, the conjugation action of $(A, \circ)$ on $(A, \cdot)$ is given by

$$(1, a) \cdot (b, 1) \cdot (1, a)^{-1} = (\lambda^{\mathrm{op}}_a(b), 1).$$

The compatibility relation

$$\rho(a)\beta(b)\rho(a)^{-1} = \beta(\lambda^{\mathrm{op}}_a(b))$$

is meant to mimic this.

Skew brace representation as group representation

- Let $A = (A, \cdot, \circ)$ be a skew brace and let (V, β, ρ) be a representation of A.
- **Observation.** The compatibility relation implies that

$$\varphi_{(\beta,\rho)} : (A, \cdot) \rtimes_{\lambda^{\mathrm{op}}} (A, \circ) \longrightarrow \mathrm{GL}(V); \ \ \varphi_{(\beta,\rho)}(a, b) = \beta(a)\rho(b)$$

  is a group representation. It is then natural to define

$$(V, \beta, \rho) \text{ has property } X \iff (V, \varphi_{(\beta,\rho)}) \text{ has property } X.$$

  Such as irreducible, indecomposable, completely reducible, etc...
- Moreover, with this observation, a lot of results from representation theory of groups can be naturally extended to representation theory of skew braces.
- **Acknowledgement.** We thank Thomas and Leandro for pointing this out.
- We realized this after we uploaded our paper onto the arXiv. Now we know that some of the results we proved follow directly from representation theory of groups and we are currently revising the paper...

Analog of Maschke's theorem and its converse

Analog of Maschke's theorem

Maschke's theorem

Let G be a finite group and let k be a field.

Assumption. The characteristic of k is either 0 or coprime to $|G|$.

Conclusion. For any finite dimensional representation (V, φ) of G over k, every φ-invariant subspace of V admits a φ-invariant complement.

Thus, every finite dimensional representation of G over k is completely reducible.

Analog of Maschke's theorem

Let A be a finite group and let k be a field.

Assumption. The characteristic of k is either 0 or coprime to $|A|$.

Conclusion. For any finite dimensional representation (V, β, ρ) of A over k, every $\varphi_{(\beta,\rho)}$-invariant subspace of V admits a $\varphi_{(\beta,\rho)}$-invariant complement.

Thus, every finite dimensional representation of A over k is completely reducible.

- The usual Maschke's theorem applies because $|(A,\cdot) \rtimes_{\lambda^{\mathrm{op}}} (A,\circ)| = |A|^2$.

Converse of Maschke's theorem

Converse of Maschke's theorem

Let G be a finite group and let k be a field.

Assumption. For any finite dimensional representation (V, φ) of G over k, every φ-invariant subspace of V admits a φ-invariant complement.

Conclusion. The characteristic of k is either 0 or coprime to $|G|$.

- **Proof.** Consider the left regular representation $(kG, \varphi_{\text{reg}})$ or G. Clearly

$$\mathrm{span}_k(\delta), \text{ where } \delta = \sum_{g \in G} g,$$

is φ_{reg}-invariant so it admits a φ_{reg}-invariant complement U.

- The subspace U has codimension one and must be equal to

$$\left\{ \sum_{g \in G} c_g g \in kG \,\middle|\, \sum_{g \in G} c_g = 0 \right\}.$$

By comparing dimensions, it suffices to prove that U lies in the above.

Converse of Maschke's theorem

- For any $\sum_{g\in G} c_g g \in kG$, the φ_{reg}-invariance of U implies that

$$\sum_{h\in G} \varphi_{\text{reg}}(h)\left(\sum_{g\in G} c_g g\right) = \sum_{h\in G}\sum_{g\in G} c_g hg = \left(\sum_{g\in G} c_g\right)\underbrace{\left(\sum_{h\in G} h\right)}_{\delta}$$

also lies in U. Since $\mathrm{span}_k(\delta)\cap U = \{0\}$, we must have

$$\sum_{g\in G} c_g = 0.$$

- To summarize, we have shown that $kG = \mathrm{span}_k(\delta)\oplus U$, where

$$\delta = \sum_{g\in G} g \text{ and } U = \left\{\sum_{g\in G} c_g g \in kG \,\middle|\, \sum_{g\in G} c_g = 0\right\}.$$

Thus $\mathrm{char}(k)$ cannot be a prime divisor of $|G|$, for otherwise $\delta\in U$. $\square$

Left regular representation of a skew brace

- Let $A = (A, \cdot, \circ)$ be a skew brace.
- Let kA be the k-vector space with the set A as a basis.
- Let $\beta_{\text{reg}} : (A, \cdot) \longrightarrow \mathrm{GL}(kA)$ be the left regular representation of $(A, \cdot)$.
- Let $\rho_{\text{reg}} : (A, \circ) \longrightarrow \mathrm{GL}(kA)$ be the left regular representation of $(A, \circ)$.

Proposition (Kozakai-T., arXiv:2405.08662)

The triplet $(kA, \beta_{\text{reg}}, \rho_{\text{reg}})$ is a representation of A.

- **Proof.** This follows from a simple calculation. □
- **Remark.** This is not the left regular representation of $(A, \cdot) \rtimes_{\lambda^{\text{op}}} (A, \circ)$!!
- Let $\beta'_{\text{reg}} : (A, \cdot) \longrightarrow \mathrm{GL}(kA)$ be the right regular representation of $(A, \cdot)$.
- Let $\rho'_{\text{reg}} : (A, \circ) \longrightarrow \mathrm{GL}(kA)$ be the right regular representation of $(A, \circ)$.
- **Remark.** In general $(kA, \beta'_{\text{reg}}, \rho'_{\text{reg}})$ is not a representation of A.

Analog of converse of Maschke's theorem

Analog of converse of Maschke's theorem

Let A be a finite group and let k be a field.

Assumption. For any finite dimensional representation (V, β, ρ) of A over k, every $\varphi_{(\beta,\rho)}$-invariant subspace of V admits a $\varphi_{(\beta,\rho)}$-invariant complement.

Conclusion. The characteristic of k is either 0 or coprime to $|A|$.

- **Proof.** Consider the left regular representation $(kA, \beta_{\text{reg}}, \rho_{\text{reg}})$ or A. Clearly

$$\mathrm{span}_k(\delta), \text{ where } \delta = \sum_{a \in A} a,$$

is $\varphi_{(\beta_{\text{reg}},\rho_{\text{reg}})}$-invariant so it admits a $\varphi_{(\beta_{\text{reg}},\rho_{\text{reg}})}$-invariant complement U.

- The exact same argument as before shows that

$$U = \left\{ \sum_{a \in A} c_a a \in kA \,\middle|\, \sum_{a \in A} c_a = 0 \right\}.$$

Thus $\mathrm{char}(k)$ cannot be a prime divisor of $|A|$, for otherwise $\delta \in U$. $\square$

Some properties of irreducible representations

Irreducible representations of a skew brace ①

- Let $A = (A, \cdot, \circ)$ be a skew brace.
- **Question.** Is it possible to classify the irreducible representations of A?
- It seems to be a very difficult question even if one has a classification of the irreducible representations of $(A, \cdot)$ and $(A, \circ)$.
- **Observation.** Let (V, β, ρ) be a representation of A.
  1. If (V, β) is irreducible, then (V, β, ρ) is also irreducible.
  2. If (V, ρ) is irreducible, then (V, β, ρ) is also irreducible.
- **Remark.** It is possible that
  1. (V, β) is reducible as a representation of $(A, \cdot)$,
  2. (V, ρ) is reducible as a representation of $(A, \circ)$,

  in an irreducible representation (V, β, ρ) of A.
- We give an explicit example on the next slides.

An explicit example ①

- **Example.** Consider the brace $(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, +, \circ)$, where

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \circ \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 + a_2 b_2 \\ a_2 + b_2 \end{pmatrix},$$

  and the multiplicative group is cyclic generated by $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$.

- Take $\beta : (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, +) \longrightarrow \mathrm{GL}_2(\mathbb{C})$ to be defined by

$$\beta \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \ \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

  The one-dimensional β-invariant subspaces are

$$\mathrm{span} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ \mathrm{span} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

  It follows that (V, β) is reducible.

An explicit example ①

- Take $\rho : (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \circ) \longrightarrow \mathrm{GL}_2(\mathbb{C})$ to be defined by
$$\rho \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$
The one-dimensional ρ-invariant subspaces are
$$\mathrm{span}\begin{pmatrix} i \\ 1 \end{pmatrix}, \ \mathrm{span}\begin{pmatrix} -i \\ 1 \end{pmatrix}.$$
It follows that (V, ρ) is reducible.
- One can verify the compatibility relation so that (V, β, ρ) is a representation of the brace in question. This is the most important part but I will omit the details...
- The representation (V, β, ρ) is irreducible because
$$\mathrm{span}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ \mathrm{span}\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \ \mathrm{span}\begin{pmatrix} i \\ 1 \end{pmatrix}, \ \mathrm{span}\begin{pmatrix} -i \\ 1 \end{pmatrix}.$$
are distinct, so there is no one-dimensional $\varphi_{(\beta,\rho)}$-invariant subspaces.

Irreducible representations of a skew brace ②

- We just saw that it is possible that both of (V,β) and (V,ρ) are reducible in an irreducible representation (V,β,ρ). Nevertheless...
- Let $A = (A,\cdot,\circ)$ be a finite skew brace.

Proposition (Kozakai-T., arXiv:2405.08662)

In an irreducible representation (V,β,ρ) of A, the representation (V,β) of the additive group $(A,\cdot)$ must be completely reducible.

- **Remark.** Essentially this holds because $(A,\cdot)$ is normal in $(A,\cdot)\rtimes_{\lambda^{\mathrm{op}}}(A,\circ)$.
- **Proof.** Since A is finite, necessarily $\dim_k(V)$ is finite.
- Hence, we a find an irreducible β-invariant subspace U of V.
- For any $a\in A$, the compatibility relation

$$\beta(\lambda_a^{\mathrm{op}}(b)) = \rho(a)\beta(b)\rho(a)^{-1}$$

and the fact that $\lambda_a^{\mathrm{op}} : A \longrightarrow A$ is a bijection imply that $\rho(a)(U)$ is also an irreducible β-invariant subspace of V.

Irreducible representations of a skew brace ②

- It follows that the non-zero subspace

$$\sum_{a\in A} \rho(a)(U)$$

  is β-invariant. But it is obviously ρ-invariant, so then it is $\varphi_{(\beta,\rho)}$-invariant.

- Since (V, β, ρ) is irreducible, we deduce that

$$V = \sum_{a\in A} \rho(a)(U).$$

  Regarding V and the summands as $k(A,\cdot)$-modules via β, we see that (V, β) must be completely reducible. $\square$

- **Fact.** Over a ring, a module that can be expressed as a sum of finitely many simple submodules is semisimple.

- **Remark.** It is possible that both of (V, ρ) is not completely reducible in an irreducible representation (V, β, ρ).

- We give an explicit example on the next slides.

An explicit example ②

- **Example.** Consider the skew brace $(S_3, \cdot, \circ)$ arising from the factorization

$$S_3 = \langle (1\,2) \rangle A_3, \ \ \langle (1\,2) \rangle \cap A_3 = 1,$$

in other words, we define $\circ$ by setting

$$\sigma \circ \tau = \sigma_1 \tau \sigma_2$$

for $\sigma, \tau \in S_3$ when $\sigma = \sigma_1 \sigma_2$ with $\sigma_1 \in \langle (1\,2) \rangle$, $\sigma_2 \in A_3$.

- The multiplicative group is $(S_3, \circ) = \langle (1\,2) \rangle \times A_3$ in this case.
- Let k be a field of characteristic 2.
- Take $\beta : (S_3, \cdot) \longrightarrow \mathrm{GL}_2(k)$ to be defined by

$$\beta(1\,2\,3) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \ \ \beta(1\,2) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

It is clear that (V, β) is irreducible.

An explicit example ②

- Take $\rho : (S_3, \circ) \longrightarrow \mathrm{GL}_2(k)$ to be defined by

$$\rho(1\,2\,3) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \;\; \rho(1\,2) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

  It is clear that (V, ρ) is not completely reducible.

- The compatibility relation holds because

$$\beta(\lambda_\sigma^{\mathrm{op}}(\tau)) = \beta(\sigma_1)\beta(\tau)\beta(\sigma_1^{-1})$$
$$\rho(\sigma)\beta(\tau)\rho(\sigma)^{-1} = \rho(\sigma_1)\beta(\tau)\rho(\sigma_1)^{-1}$$

  and they are equal because β, ρ agree on $\langle(1\,2)\rangle$.

- Clearly (V, β, ρ) is irreducible because (V, β) is irreducible.
- Clearly (V, ρ) is not completely reducible because $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ is not diagonalizable.

Final remark

- Can one apply skew brace representation to get some structural results about skew braces?

- Perhaps the definition of Zhu and Letourmy-Vendramin is not so useful in this aspect? If that is the case, what would be a “better” definition of skew brace representation?

References (by order of appearance)

1. Guarnieri, L.; Vendramin, L. Skew braces and the Yang-Baxter equation.
   Math. Comp. 86 (2017), no. 307, 2519–2534.
2. Jespers, E.; Kubat, Ł.; Van Antwerpen, A.; Vendramin, L. Factorizations of skew braces.
   Math. Ann. 375 (2019), no. 3-4, 1649–1663.
3. Tsang, Cindy. A generalization of Ito's theorem to skew braces.
   J. Algebra 642 (2024), 367–399.
4. Letourmy, T.; Vendramin, L. Isoclinism of skew braces.
   Bull. Lond. Math. Soc. 55 (2023), no. 6, 2891–2906.
5. Caranti, A. Some properties of skew braces that are invariant under isoclinism.
   Comm. Algebra 52 (2024), no. 3, 1118–1123.
6. Letourmy, T.; Vendramin, L. Schur covers of skew braces.
   J. Algebra 644 (2024), 609–654.
7. Zhu, Haixing. The construction of braided tensor categories from Hopf braces.
   Linear Multilinear Algebra 70 (2022), no. 16, 3171–3188.

Thank you for listening!